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One generalization of the second Painlevé hierarchy

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Abstract
New integrable hierarchy of ordinary differential equations is presented. This hierarchy is shown to generalize the P_2 hierarchy, while on the other hand, this hierarchy gives a special case of the third Painlevé equation. The isomonodromic linear problem is used to find this new hierarchy. The Painlevé approach to investigate the fourth-order ordinary differential equation of this hierarchy is applied.

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1. Introduction

In 1884 L Fuchs and A Poincaré stated the following problem: define new functions by means of ordinary differential equations (ODEs), necessarily nonlinear [1]. This problem was solved by P Painlevé and his school who studied the second-order ODE class of certain form and found 50 types of canonical equations whose solutions have no movable critical points. They also found that among these 50 equations, 44 are solvable in terms of previously known functions (such as elliptic functions and solutions of linear equations). Painlevé and his collaborators also showed that there are exactly six second-order ordinary differential equations defining new functions. At present these functions are called Painlevé transcendents, and equations with general solutions in the form of transcendents are called Painlevé equations. Six Painlevé equations were first discovered from a strictly mathematical consideration, but these equations have recently appeared in several physical applications [2].

The results of Painlevé and his collaborators led to the problem of finding other new functions that could be defined by nonlinear ODEs such as the Painlevé transcendents. Attempts along these lines were undertaken in works [3, 4], where four hierarchies of ordinary differential equations were presented to find general solutions which are transcendental functions with respect to constants of integration. In addition to the Painlevé equations, four fourth-order ODEs of these hierarchies were proved to have general solutions in the form of transcendents [5, 6].

The aim of this paper is to present a new hierarchy of ordinary differential equations which takes the form

$$\varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z} + y_z\right) L^m\left[y_{zz} - \frac{1}{2}y_z^2\right] + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}(zy_z) + \beta\exp(y) + \delta\exp(-y) = 0 \qquad (1.1)$$

where $y(z)$ is an unknown function, β, δ and ε_0 are parameters and the operator L^m is determined by the Lenard relation using the following formulae:

$$\frac{\mathrm{d}}{\mathrm{d}z}L^{m+1} = L^m_{zzz} + 2yL^m_z + y_z L^m \qquad L^0[y] = 1. \qquad (1.2)$$

It should be noted that if we assume $\beta = \delta = 0$ and $v = y_z$ in equation (1.1), we obtain the P_2 hierarchy after integration,

$$\varepsilon_0\left(\frac{\mathrm{d}}{\mathrm{d}z} + v\right) L^m\left[v_z - \frac{1}{2}v^2\right] + \frac{1}{2}zv - \alpha = 0. \qquad (1.3)$$

Here α is a parameter obtained from integration of equation. On the other hand, equation (1.1) gives a special case of the third Painlevé equation because assuming $\varepsilon_0 = 0$ in equation (1.1) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}(zy_z) + \beta\exp(y) + \delta\exp(-y) = 0. \qquad (1.4)$$

If we take $y = \ln v$ we obtain from equation (1.4) a special case of the third Painlevé equation,

$$v_{zz} - \frac{v_{zz}^2}{v} + \frac{1}{z}v_z + \frac{2}{z}(\beta v^2 + \delta) = 0. \qquad (1.5)$$

Solutions of equations (1.3) and (1.5) are known to be the transcendental functions with respect to constants of integration. However, equations (1.3) and (1.5) are special cases of hierarchy (1.1). Consequently, solutions of equation (1.1) are transcendental functions with respect to constants of integration too.

Assuming $m = 1$ in equation (1.1), we have the fourth-order ordinary differential equation of the form

$$\varepsilon_0 y_{zzzz} + \tfrac{1}{2}\left(z - 3\varepsilon_0 y_z^2\right) y_{zz} + \tfrac{1}{2}y_z + \beta\exp(y) + \delta\exp(-y) = 0. \qquad (1.6)$$

This equation is a generalization of the second and third Painlevé equations. The general solution of equation (1.6) is expressed by the transcendental function with respect to constants of integration too. If we prove the irreducibility of equation (1.6) we will obtain that the general solution of equation (1.6) as a generalization of transcendents for the second and third Painlevé equations will be the new transcendental function with respect to constants of integration. In other words, we hope that the general solution of equation (1.6) is different from the Painlevé transcendents. It is important to study the general solution of this equation because these transcendents can be special solutions of nonlinear partial differential equations or they can be used in the description of some nonlinear mathematical models. We are going to study this in this paper.

The outline of this paper is as follows. Using the isomonodromic linear problem, the new hierarchy of ordinary differential equations is found in section 2. In this section we have the Lax pairs to solve this hierarchy. The application of the Painlevé test for investigation of the hierarchy of fourth-order ordinary differential equations is discussed in section 3.

2. Isomonodromic linear problem for equation (1.1)

In this section we are going to use the isomonodromic linear problem to look for the hierarchy of ordinary differential equation (1.1).

Let us assume that

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{2.1}$$

is a solution of the linear system of equations [2]

$$\Psi_z = M\Psi \tag{2.2}$$

$$\lambda^2 \Psi_\lambda = N\Psi \tag{2.3}$$

where M and N are 2×2 matrices and λ is a spectral parameter. The compatibility condition of equations (2.2) and (2.3),

$$(\Psi_z)_\lambda = (\Psi_\lambda)_z \tag{2.4}$$

takes the form of the matrix equation

$$\lambda^2 M_\lambda + MN = NM + N_z. \tag{2.5}$$

Let us take matrices M and N of the form

$$M = \begin{pmatrix} -\mathrm{i}\lambda & \frac{1}{2}y_z \\ \frac{1}{2}y_z & \mathrm{i}\lambda \end{pmatrix} \qquad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{2.6}$$

Equation (2.5) can be presented in the form of the system of equations

$$A_z + \tfrac{1}{2}y_z(B - C) + \mathrm{i}\lambda^2 = 0 \tag{2.7}$$

$$B_z + 2\mathrm{i}\lambda B + Ay_z = 0 \tag{2.8}$$

$$C_z - 2\mathrm{i}\lambda C + Ay_z = 0. \tag{2.9}$$

Let us look for elements A, B and C of the form

$$A(y, \ldots, \lambda) = \sum_{k=0}^{2n} a_k \lambda^{2n-k} \tag{2.10}$$

$$B(y, \ldots, \lambda) = \sum_{k=0}^{2n} b_k \lambda^{2n-k} \tag{2.11}$$

$$C(y, \ldots, \lambda) = \sum_{k=0}^{2n} c_k \lambda^{2n-k}. \tag{2.12}$$

Here $n = 1, 2, \ldots$. Substituting equations (2.10)–(2.12) into equations (2.7)–(2.9), we have the following set of equations:

$$b_0 = c_0 = 0 \tag{2.13}$$

$$a_{k,z} + \tfrac{1}{2}(b_k - c_k)y_z = 0 \qquad k = 0, \ldots, 2n-3, 2n-1, 2n \tag{2.14}$$

$$2\mathrm{i}b_k + b_{k-1,z} + a_{k-1}y_z = 0 \qquad k = 1, 2, \ldots, 2n \tag{2.15}$$

$$2\mathrm{i}c_k - c_{k-1,z} + a_{k-1}y_z = 0 \qquad k = 1, 2, \ldots, 2n \tag{2.16}$$

$$a_{2n-2,z} + \tfrac{1}{2}(b_{2n-2} - c_{2n-2})y_z + \mathrm{i} = 0 \tag{2.17}$$

$$b_{2n,z} + a_n y_z = 0 \tag{2.18}$$

$$c_{2n,z} - a_n y_z = 0. \tag{2.19}$$

These equations can be solved sequentially step by step.

Substituting equations (2.13) into equations (2.14), we have

$$a_0 = -(2\mathrm{i})^{2n-1}\varepsilon_0. \tag{2.20}$$

Using (2.20) and equations (2.15) and (2.16), we get

$$b_1 = c_1 = (2\mathrm{i})^{2n-2}\varepsilon_0 y_z \qquad a_1 = 0 \tag{2.21}$$

From equations (2.14)–(2.16) one can also find

$$b_2 = -c_2 = -(2\mathrm{i})^{2n-3}\varepsilon_0 y_{zz} \tag{2.22}$$

and

$$a_2 = \mathrm{i}\varepsilon_0(2\mathrm{i})^{2n-4} y_z^2. \tag{2.23}$$

Taking into account these solutions we have from equations (2.14)–(2.16), by the method of mathematical induction, the following formulae:

$$b_{2l} = -c_{2l} = -\varepsilon_0(2\mathrm{i})^{2n-2l-1}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z} + y_z\right) L^{l-1}\left[y_{zz} - \frac{1}{2}y_z^2\right] \tag{2.24}$$

$$a_{2l,z} = -b_{2l}y_z \qquad l = 1, 2, \ldots, n-2 \tag{2.25}$$

$$b_{2l+1} = c_{2l+1} = \varepsilon_0(2\mathrm{i})^{2n-2l-2}\left(\frac{\mathrm{d}}{\mathrm{d}z} + y_z\right) L^{l}\left[y_{zz} - \frac{1}{2}y_z^2\right] \tag{2.26}$$

$$a_{2l+1} = 0 \qquad l = 1, 2, \ldots, n-2. \tag{2.27}$$

From equations (2.17) and (2.14) we have as well

$$b_{2n-2} = -c_{2n-2} = -2\mathrm{i}\varepsilon_0\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z} + y_z\right) L^{n-2}\left[y_{zz} - \frac{1}{2}y_z^2\right] \tag{2.28}$$

$$a_{2n-2,z} = -\mathrm{i} - b_{2n-2}y_z \tag{2.29}$$

$$b_{2n-1} = c_{2n-1} = \frac{1}{2}zy_z + \varepsilon_0\left(\frac{\mathrm{d}}{\mathrm{d}z} + y_z\right) L^{n-1}\left[y_{zz} - \frac{1}{2}y_z^2\right] \tag{2.30}$$

$$a_{2n-1} = 0. \tag{2.31}$$

At $k = 2n$ we have solutions from equations (2.16), (2.18) and (2.19) of the form

$$a_{2n} = \frac{\mathrm{i}}{2}(\beta\exp(y) - \delta\exp(-y)) \tag{2.32}$$

$$b_{2n} = -\frac{\mathrm{i}}{2}(\beta\exp(y) + \delta\exp(-y)) \tag{2.33}$$

$$c_{2n} = \frac{\mathrm{i}}{2}(\beta\exp(y) + \delta\exp(-y)). \tag{2.34}$$

Equations (2.15) and (2.16) at $k = 2n-1$ give an equation of the form

$$\varepsilon_0\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z} + y_z\right) L^{n-1}\left[y_{zz} - \frac{1}{2}y_z^2\right] + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}(zy_z) + \beta\exp(y) + \delta\exp(-y) = 0. \tag{2.35}$$

This is exactly the hierarchy (1.1) at $m = n-1$. Thus, using the isomonodromic linear problem (2.2), we have found equation (1.1).

As a byproduct of the isomonodromic linear problem (2.2), we can use this one to solve equation (1.1).

Let us present solutions of equations (2.13)–(2.19) at $n = 2$. In this case we have

$$a_0 = 8\mathrm{i}\varepsilon_0 \tag{2.36}$$

$$b_1 = c_1 = -4\varepsilon_0 y_z \qquad a_1 = 0 \tag{2.37}$$

$$b_2 = -c_2 = -2\mathrm{i}\varepsilon_0 y_{zz} \qquad a_2 = -\mathrm{i}z + \mathrm{i}\varepsilon_0 y_z^2 \tag{2.38}$$

$$b_3 = c_3 = \tfrac{1}{2}zy_z + \varepsilon_0\left(y_{zzz} - \tfrac{1}{2}y_z^3\right). \tag{2.39}$$

Coefficients b_4, c_4 and a_4 are determined from formulae (2.32)–(2.34). Equations (2.15) and (2.16) give the fourth-order ordinary differential equation (1.6).

In the case $n = 3$ we have from equations (2.20) and (2.31), the following formulae:

$$a_0 = -32\mathrm{i}\varepsilon_0 \qquad b_1 = c_1 = 16\varepsilon_0 y_z \qquad a_1 = 0 \tag{2.40}$$

$$b_2 = -c_2 = 8\mathrm{i}\varepsilon_0 y_{zz} \qquad a_2 = -4\mathrm{i}\varepsilon_0 y_z^2 \tag{2.41}$$

$$b_3 = c_3 = -4\varepsilon_0\left(y_{zzz} - \tfrac{1}{2}y_z^3\right) \qquad a_3 = 0 \tag{2.42}$$

$$b_4 = -c_4 = -2\mathrm{i}\varepsilon_0\left(y_{zzzz} - \tfrac{3}{2}y_z^2 y_{zz}\right) \tag{2.43}$$

$$a_4 = -\mathrm{i}z + 2\mathrm{i}\varepsilon_0 y_z y_{zzz} - \mathrm{i}\varepsilon_0 y_{zz}^2 - \tfrac{3}{4}\mathrm{i}\epsilon_0 y_z^4 \tag{2.44}$$

$$b_5 = c_5 = \tfrac{1}{2}zy_z + \varepsilon_0 y_{zzzzz} - \tfrac{5}{2}\varepsilon_0 y_z^2 y_{zzz} - \tfrac{5}{2}\varepsilon_0 y_z y_{zz}^2 + \tfrac{3}{8}\varepsilon_0 y_z^5 \qquad a_5 = 0. \tag{2.45}$$

Coefficients b_6, c_6 and a_6 are determined from formulae (2.32)–(2.34). We also have the sixth-order ordinary differential equation that corresponds to equation (1.1) at $m = 2$.

The linear system (2.2), (2.3) provides the key to solving equation (1.1) for initial data by the inverse monodromy method.

3. Painlevé test for equation (1.1) at $m = 1$

The Painlevé test is a powerful method to investigate the integrability of differential equations. This approach allows one to find the necessary conditions for the absence of movable critical singularities in the general solution of a differential equation. Let us apply this approach to equation (1.1) at $m = 1$, $\beta = 72\gamma$, $\delta = -72\,\mu$ and $\varepsilon_0 = 1$. Here γ and μ are arbitrary constants. First, we need to transform this equation into a polynomial form. Assuming $y = \ln u$, we get the following equation:

$$\begin{aligned} &u^3 u_{zzzz} - 4u^2 u_z u_{zzz} + \tfrac{21}{2} u u_z^2 u_{zz} - 3u^2 u_{zz}^2 - \tfrac{9}{2} u_z^4 + 72\gamma u^5 - 72\mu u^3 - \tfrac{1}{2} z u^3 u_{zz} \\ &\qquad + \tfrac{1}{2} z u^2 u_z^2 - \tfrac{1}{2} u^3 u_z = 0. \end{aligned} \tag{3.1}$$

To study equation (3.1) in the Painlevé test, we use the perturbative Painlevé approach presented in [1, 7].

Substituting

$$u = \frac{u_0}{x^p} \qquad x = z - z_0 \tag{3.2}$$

into leading members of equation (3.1), we get two families of solutions with values $(p, u_0) = (-4, 1/\gamma)$ and $(p, u_0) = (4, \mu)$. One can see that the first necessary condition for the absence of movable critical singularities of the logarithmic type is satisfied.

To study the second necessary condition, we look for the Fuchs indices. At this stage we assume

$$u = \frac{1}{\gamma x^4} + u_j x^{j-4} \qquad x = z - z_0 \tag{3.3}$$

and substitute equation (3.3) into leading members of equation (3.1) again. We have the following Fuchs indices: $(-3, -1, 4, 6)$.

Assuming for the second family

$$u = \mu x^4 + u_j x^{j+4} \qquad x = z - z_0 \tag{3.4}$$

we find the same Fuchs indices $(-3, -1, 4, 6)$ in this case. We have obtained that the second necessary condition is satisfied too.

The third necessary condition corresponds to checking the existence of the Laurent series for the general solution of equation (3.1). We have the negative indices for the two families of solutions and we want to look for solutions of equation (3.1) of the form [7]

$$u = \sum_{k=0}^{\infty} \varepsilon^k u^{(k)}. \tag{3.5}$$

Equation (3.1) in this case takes the form

$$E\,(z, u) = \sum_{k=0}^{\infty} \varepsilon^k E^{(k)}. \tag{3.6}$$

We have

$$k = 0\colon E^{(0)} \equiv E'\big(x, u^{(0)}\big) = 0 \tag{3.7}$$

$$k = 1\colon E^{(1)} \equiv E'\big(x, u^{(0)}\big)\, u^{(1)} = 0 \tag{3.8}$$

$$k \geqslant 2\colon E^{(k)} \equiv E'\big(x, u^{(0)}\big)\, u^{(k)} + R^{(k)}\big(x, u^{(0)}, \ldots, u^{(k-1)}\big) = 0 \tag{3.9}$$

where E' is the Gateaux derivative of equation (3.1) and $R^{(k)}$ corresponds to the contribution of previous members of the expansions [7].

The components of solution $u^{(k)}$ are looked for in terms of the Laurent series

$$u^{(0)} = \sum_{j=0}^{\infty} u_j^{(0)} x^{j-4} \qquad u^{(1)} = \sum_{j=-3}^{\infty} u_j^{(1)} x^{j-4} \qquad \ldots \qquad u^{(k)} = \sum_{j=-3k}^{\infty} u_j^{(k)} x^{j-4}. \tag{3.10}$$

The coefficients $u_j^{(k)}$, $j = 0, 1, \ldots$, are found after the substitution of equations (3.10) into equations (3.7)–(3.9) and equating $E^{(k)}$ to zero. The absence of movable critical points corresponds to members $E_j^{(k)} = 0$, $k = 0, 1, \ldots$, and arbitrary coefficients $u_r^{(k)}$, where r are the Fuchs indices. As this takes place the arbitrary values $u_r^{(k)}$ are introduced at $k = 0$ for $r \geqslant 0$ and at $k \geqslant 1$ for $r \leqslant -1$.

We have obtained the following expression for $u^{(0)}$:

$$u^{(0)} = \frac{1}{\gamma (z - z_0)^4} + \frac{3z_0}{5\gamma (z - z_0)^2} + a_4 - \frac{9z_0}{10\gamma}(z - z_0) + u_6 (z - z_0)^2 + \cdots \tag{3.11}$$

where z_0, u_4 and u_6 are arbitrary constants corresponding to the positive Fuchs indices and -1. We have used the expansion equation (3.11) to check the negative Fuchs indices for the first family of solutions of equation (3.1). It has been found that this family of solutions satisfies the third necessary condition for the absence of movable critical points for $0 \leqslant k \leqslant 4$. In this case we obtained four arbitrary constants in the Laurent series.

For the second family of solutions, we have

$$u^{(0)} = \mu (z - z_0)^4 - \frac{3\mu}{5} z_0 (z - z_0)^6 + u_4 (z - z_0)^8 + \frac{9\mu}{10} z_0 (z - z_0)^9 + u_6 (z - z_0)^{10} + \cdots \tag{3.12}$$

where z_0, u_4 and u_6 are arbitrary constants. Taking into account this expansion we found four arbitrary constants in the Laurent series for solution of the second family at $0 \leqslant k \leqslant 4$ too.

Thus equation (3.1) has two families of solutions with negative Fuchs indices, and the perturbative Painlevé approach shows that equation (3.1) passes the Painlevé test.

4. Conclusion

Let us emphasize in brief the results of this work. We have introduced the new hierarchy of ODEs. On one hand this hierarchy is a generalization of the second Painlevé hierarchy, and on the other, this hierarchy generalizes a special case of the third Painlevé equation. We have shown how to look for the new hierarchies using the isomonodromic linear problems. As an example, we used the isomonodromic linear problem of the special form for the third Painlevé equation. The linear system of equations found can be used for the description of the solution for initial data by the inverse monodromy transform. The general solutions of equations for this hierarchy were shown to be the transcendental functions with respect to constants of integration, and we hope these transcendents will be new because these are the generalizations of the second Painlevé hierarchy and the third Painlevé equation. We think that it is important to study these transcendents because they can be special solutions of nonlinear partial differential equations. However, for the time being we need to prove irreducibility for equations of this hierarchy. We also need to look for the birational transformations, rational solutions and other properties for this hierarchy in future.

Application of the Painlevé approach to the fourth-order ordinary differential equation of the hierarchy studied was given. As expected, this equation passes the Painlevé test. The local representations of the general solution for the equation considered were presented.

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